Section 5.4: Exponential & Logarithmic Models

exponential model: \( y = a \cdot b^{kx} = a \cdot (b^k)^x = a \cdot B^x \)
Insurance Premiums

The monthly premiums for $250,000 in term-life insurance over a 10-year term period increase with the age of the men purchasing the insurance. The monthly premiums for nonsmoking males are shown in Table 5.4.

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>Monthly Premium for 10-Year Term Insurance (dollars)</th>
<th>Age (years)</th>
<th>Monthly Premium for 10-Year Term Insurance (dollars)</th>
</tr>
</thead>
<tbody>
<tr>
<td>35</td>
<td>123</td>
<td>60</td>
<td>783</td>
</tr>
<tr>
<td>40</td>
<td>148</td>
<td>65</td>
<td>1330</td>
</tr>
<tr>
<td>45</td>
<td>225</td>
<td>70</td>
<td>2448</td>
</tr>
<tr>
<td>50</td>
<td>338</td>
<td>75</td>
<td>4400</td>
</tr>
<tr>
<td>55</td>
<td>500</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(Source: Quotesmith.com)

a. Graph the data in the table with $x$ as age and $y$ in dollars.

b. Create an exponential function that models these premiums as a function of age.

c. Graph the data and the exponential function that models the data on the same axes.

$$f(x) \approx 4.039 \cdot 1.095^x$$
Technology Note

When using technology to fit an exponential model to data, you should align the inputs to reasonably small values.

Q: How can you tell if an exponential model fits the data exactly or only approximately?

A: If $R^2$-value in Graph is 1, then probably an exact fit; otherwise, an approximate fit.
Constant Percent Changes

If the percent change of the outputs of a set of data is constant for equally spaced inputs, an exponential function will be a perfect fit for the data.

If the percent change of the outputs is approximately constant for equally spaced inputs, an exponential function will be an approximate fit for the data.

\[
\text{percent change} = \frac{\text{amount of change}}{\text{original}}
\]

(Ex) If something cost $5 yesterday and $6 today, the percent change is

\[
\% \text{ change} = \frac{6-5}{5} = \frac{1}{5} \implies 20\% \text{ change}
\]
|
|---|---|---|---|---|---|
| **$x$** (hours) | 0  | 1  | 2  | 3  | 4  |
| **$y$** (paramecia) | 1  | 2  | 4  | 8  | 16 |

First differences (change in $y$-coords):

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$2$</td>
<td>$4$</td>
<td>$8$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

% Change:

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$100%$</td>
<td>$100%$</td>
<td>$100%$</td>
<td>$100%$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The % changes are constant, so an exponential model will fit this data exactly.
Sales Decay

Suppose a company develops a product that is released with great expectations and extensive advertising, but sales suffer because of bad word of mouth from dissatisfied customers.

a. Use the monthly sales data shown in Table 5.13 to determine the percent change for each of the months given.

b. Find the exponential function that models the data.

c. Graph the data and the model on the same axes.

Table 5.13

<table>
<thead>
<tr>
<th>Month</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>780</td>
<td>608</td>
<td>475</td>
<td>370</td>
<td>289</td>
<td>225</td>
<td>176</td>
<td>137</td>
</tr>
</tbody>
</table>

1st diffs: 
-172 → -133 → -81 → -64 → -49 → -39

% Change:
-172/780 = -22.05%
-133/608 = -21.88%

The data can be fit approximately by an exponential model.
Logarithmic models: \( f(x) = a + b \ln x \)  \( (b > 0, x > 0) \)

**Technology Note**

When using technology to fit a logarithmic model to data, you must align the data so that all input values are positive.
Female Workers

The percent of all workers who are female during selected years from 1970 to 2006 is given in Table 5.12.

a. Find a logarithmic function that models these data. Align the input to be the number of years after 1960.

b. Graph the equation and the aligned data points. Comment on how the model fits the data.

c. Assuming that the model is valid in 2010, use it to estimate the percent of female workers in 2010.

\[ f(x) \approx 11.244 + 7.978 \ln x \]

\[ \text{with } R^2 \approx 0.97 \]

c) In 2010, \( x = 2010 - 1960 = 50 \)

Find \( f(50) \)!

<table>
<thead>
<tr>
<th>Year</th>
<th>Percent of Workers Who Are Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>1970</td>
<td>30.0</td>
</tr>
<tr>
<td>1975</td>
<td>31.9</td>
</tr>
<tr>
<td>1980</td>
<td>35.3</td>
</tr>
<tr>
<td>1985</td>
<td>37.9</td>
</tr>
<tr>
<td>1990</td>
<td>37.2</td>
</tr>
<tr>
<td>1995</td>
<td>40.3</td>
</tr>
<tr>
<td>2000</td>
<td>41.2</td>
</tr>
<tr>
<td>2005</td>
<td>41.3</td>
</tr>
<tr>
<td>2006</td>
<td>41.5</td>
</tr>
</tbody>
</table>