Section 5.3: Solving Exponential and Logarithmic Equations

**Change of Base Formula**

If \( b > 0, b \neq 1, a > 0, a \neq 1, \) and \( x > 0, \) then

\[
\log_a x = \frac{\log_b x}{\log_b a}
\]

In particular, for base 10 and base \( e, \)

\[
\log_a x = \frac{\log x}{\log a} \quad \text{and} \quad \log_a x = \frac{\ln x}{\ln a}
\]

**Example**

\[
\log_{23} 947 = \frac{\log 947}{\log 23}
\]

**Approximate** \( \log_{23} 947 \) to 2 decimal places.

1. \( \log_{23} 947 = \frac{\log 947}{\log 23} \approx 2.19 \)

2. \( \log_{23} 947 = \frac{\ln 947}{\ln 23} \approx 2.19 \)
Applying the Change of Base Formula

a. Evaluate \( \log_8 124 \).

Graph the functions in parts (b) and (c) by changing each logarithm to a common logarithm and then by changing the logarithm to a natural logarithm.

b. \( y = \log_3 x \)  

c. \( y = \log_2 (-3x) \)

\[ \text{a) We know that } 2 < \log_8 124 < 3 \text{ b/c } 8^2 = 64 \text{ and } 8^3 = 512. \]

\[ \log_8 128 = \frac{\ln 124}{\ln 8} \approx 2.318 \]

\[ \text{b)} \quad y = \log_3 x = \log_b (x, 3) = \frac{\log (x)}{\log (3)} = \frac{\ln (x)}{\ln (3)} \]

\[ \text{c)} \quad y = \log_2 (-3x) = \log_b (-3x, 2) = \frac{\log (-3x)}{\log (2)} = \frac{\ln (-3x)}{\ln (2)} \]
Solving a Base 10 Exponential Equation

a. Solve the equation $3000 = 150(10^{4t})$ for $t$ by converting it to logarithmic form.

b. Solve the equation graphically to confirm the solution.

\[
\begin{align*}
a) & \quad 150 \cdot 10^{4t} = 3000 \quad \Rightarrow \quad 10^{4t} = \frac{3000}{150} = 20 \\
& \quad \Rightarrow \quad 10^{4t} = 20 \\
& \quad \ln 10^{4t} = \ln 20 \\
& \quad 4t \ln 10 = \ln 20 \\
& \quad (4\ln 10) t = \ln 20 \\
& \quad t = \frac{\ln 20}{4\ln 10} \quad \text{(exact)} \\
& \quad = \frac{\ln 20}{\ln 10000} \quad \text{(exact)} \approx 0.325 \quad \text{(approx)} \\
\end{align*}
\]
Solution of Exponential Equations

Solve the following exponential equations.

a. \(4096 = 8^{2x}\)

b. \(6(4^{3x-2}) = 120\)

\[
a) \quad 8^{2x} = 4096 \\
\ln 8^{2x} = \ln 4096 \\
2 \cdot \ln 8 = \ln 4096 \\
x = \frac{\ln 4096}{2 \ln 8} \\
= 2 \\
\text{check: } 8^{2(2)} = 4096 \quad \sqrt
\]

\[
b) \quad 6 \left(4^{3x-2}\right) = 120 \\
\frac{4^{3x-2}}{6} = 20 \\
\ln 4^{3x-2} = \ln 20 \\
(3x-2) \ln 4 = \ln 20 \\
3x-2 = \frac{\ln 20}{\ln 4} \\
x = \frac{\ln 20}{3 \ln 4} + 2 \\
x \approx 1.387
\]
Carbon-14 Dating

Radioactive carbon-14 decays according to the equation

\[ y = y_0 e^{-0.00012097t} \]

where \( y_0 \) is the original amount and \( y \) is the amount of carbon-14 at time \( t \) years. To find the age of a fossil if the original amount of carbon-14 was 1000 grams and the present amount is 1 gram, we solve the equation

\[ 1 = 1000e^{-0.00012097t} \]

\ a. \ Find the age of the fossil by converting the equation to logarithmic form.

\ b. \ Find the age of the fossil using graphical methods.

\ a) \ Solve \ 1000e^{-0.00012097t} = 1 \ for \ t.

\[ e^{-0.00012097t} = \frac{1}{1000} \]

\[ \ln e^{-0.00012097t} = \ln 0.001 \]

\[ -0.00012097t \ln e = \ln 0.001 \]

\[ t = \frac{\ln 0.001}{-0.00012097} \approx 57,103 \ \text{years old}. \]
Investment

If $10,000 is invested for \( t \) years at 10\% compounded annually, the future value is given by

\[
S = P(1+r)^t
\]

In how many years will the investment grow to $45,950?

Solve

\[
\frac{10,000 (1.10^t)}{10,000} = \frac{45,950}{10,000}
\]

\[
1.10^t = 4.595
\]

\[
\ln (1.10^t) = \ln 4.595
\]

\[
t \ln 1.10 = \ln 4.595
\]

\[
t = \frac{\ln 4.595}{\ln 1.10} \quad \text{(exact)}
\]

\[\approx 16 \text{ years}\]
\[ 6 + 9^{2x+7} = 47 \]

\[ \frac{6 + 9^{2x+7} - 6}{1} = 41 \]

\[ 9^{2x+7} = 41 \]

\[ \ln 9^{2x+7} = \ln 41 \]

\[ (2x+7) \ln 9 = \ln 41 \]

\[ \frac{(2x+7)}{\ln 9} = \frac{\ln 41}{\ln 9} \]

\[ 2x + 7 = \frac{\ln 41}{\ln 9} \]

\[ 2x = \frac{\ln 41}{\ln 9} - 7 \]

\[ x = \frac{\frac{\ln 41}{\ln 9} - 7}{2} \text{ (exact)} \]
**Doubling Time**

a. Prove that the time it takes for an investment to double its value is \( t = \frac{\ln 2}{r} \) if the interest rate is \( r \), compounded continuously. \( S = Pe^{rt} \)

b. Suppose $2500 is invested in an account earning 6% annual interest, compounded continuously. How long will it take for the amount to grow to $5000?

\[ a) \text{ If you start with } P \text{ dollars and that amount doubles, what will you end with? end with } 2P \text{ dollars (future value)} \]

Solve \( Pe^{rt} = \frac{2P}{P} \)

\[ e^{rt} = 2 \]

\[ rt \ln e = \ln 2 \]

\[ rt = \ln 2 \]

\[ t = \frac{\ln 2}{r} \]

b) Compute \( \frac{\ln 2}{0.06} \approx 11.5 \text{ yrs} \)
Solving a Logarithmic Equation

Solve $4 \log_3 x = -8$ by converting to exponential form and verify the solution graphically.

\[
4 \log_3 x = -8 \\
\log_3 x = -2 \quad \text{(log form of a logarithm)} \\
3^{-2} = x \quad \text{(exp form of a logarithm)} \\
x = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}
\]
Solving a Logarithmic Equation

a. Solve $6 + 3 \ln x = 12$ by writing the equation in exponential form.

b. Solve the equation graphically.

\[ -6 + 3 \ln x = 12 \]

\[ 3 \ln x = 6 \]

\[ \ln x = 2 \]

\[ e^2 = x \quad \Rightarrow \quad x = e^2 \quad (\text{exact value}) \]

\[ \approx 7.39 \quad (\text{approx}) \]
Solving a Logarithmic Equation

Solve \( \ln x + 3 = \ln(x + 4) \) by converting the equation to exponential form and then using algebraic methods.

\[
\ln x + 3 = \ln(x + 4)
\]

\[
3 = \ln(x + 4) - \ln x
\]

\[
\ln \left( \frac{x + 4}{x} \right) = 3
\]

\[
\frac{x + 4}{x} = e^3 \quad \text{(assume } x \neq 0)\]

\[
x + 4 = xe^3
\]

\[
x - xe^3 = -4
\]

\[
x \left(1 - e^3\right) = -4 \quad \implies \quad x = \frac{-4}{1 - e^3} = \frac{4}{e^3 - 1} \approx 0.21
\]
Global Warming

In an effort to reduce global warming, it has been proposed that a tax be levied based on the emissions of carbon dioxide into the atmosphere. The cost–benefit equation \( \ln(1 - P) = -0.0034 - 0.0053t \) estimates the relationship between the percent reduction of emissions of carbon dioxide \( P \) (as a decimal) and the tax \( t \) in dollars per ton of carbon.

(Source: W. Clime, *The Economics of Global Warming*)

**a.** Solve the equation for \( P \), the estimated percent reduction in emissions.

**b.** Determine the estimated percent reduction in emissions if a tax of $100 per ton is levied.

\[
a) \quad \ln(1 - P) = -0.0034 - 0.0053t \\
\quad 1 - P = e^{-0.0034 - 0.0053t} \\
\quad P = 1 - e^{-0.0034 - 0.0053t} \\
\[
\]

**b** Compute

\[
P = 1 - e^{-0.0034 - 0.0053(100)} \approx 41.3\% 
\]
Ex. Solve \( \log_3 x + \log_3 (x-2) = 2 \)

\[
\begin{align*}
\log_3 x(x-2) & = 2 \\
\log_3 (x^2 - 2x) & = 2 \\
\end{align*}
\]

\[
\begin{align*}
x^2 - 2x & = 3^2 = 9 \\
x^2 - 2x - 9 & = 0
\end{align*}
\]

Use Quadratic Formula:

\[
x = \frac{1 \pm \sqrt{10}}{2} = 1 + \sqrt{10} \text{ or } 1 - \sqrt{10}
\]

outside the domain of \( \log_3 x \)