Math 3005 – Section 5.2 Homework

5.7 Prove that there is no largest negative rational number.

Proof. Assume, to the contrary, that there is a largest negative rational number; call it \( r \). Then \( 2r \), which is the product of two rational numbers, is a negative rational number. Since \( 2r \) is farther from 0 than \( r \) is, we have that \( 2r < r \), which implies that \( r < r/2 \). Now, \( r/2 = \frac{1}{2}r \) is the product of two rational numbers, so \( r/2 \) a negative rational number that is larger than \( r \), resulting in a contradiction. \( \square \)

5.16 Prove that \( \sqrt{3} \) is irrational.

Before we begin the proof, we note the following lemma, which was proven in the homework for Chapter 4, Exercise 4.3.

Lemma. Let \( m \in \mathbb{Z} \). Then \( 3 \mid m \) if and only if \( 3 \mid m^2 \).

Proof. Assume, to the contrary, that \( \sqrt{3} \) is rational. Then \( \sqrt{3} = p/q \) for some integers \( p \) and \( q \), where \( q \neq 0 \) and \( p \) and \( q \) have no common factors. Thus, \( p^2 = 3q^2 \), which implies that \( 3 \mid p^2 \). By the lemma above, we conclude that \( 3 \mid p \); that is, \( p = 3k \) for some integer \( k \). Now, \( 3q^2 = (3k)^2 \) implies that \( q^2 = 3k^2 \), or equivalently, that \( 3 \mid q^2 \). Again, by the lemma, we conclude that \( 3 \mid q \). Since \( 3 \mid p \) and \( 3 \mid q \), the integers \( p \) and \( q \) have a common factor, resulting in a contradiction. \( \square \)