Odd-Graceful Labelings of Trees of Diameter 5

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Abstract

A difference vertex labeling of a graph $G$ is an assignment $f$ of labels to the vertices of $G$ that induces for each edge $xy$ the weight $|f(x) - f(y)|$. A difference vertex labeling $f$ of a graph $G$ of size $n$ is odd-graceful if $f$ is an injection from $V(G)$ to $\{0, 1, \ldots, 2n - 1\}$ such that the induced weights are $\{1, 3, \ldots, 2n - 1\}$. We show here that any forest whose components are caterpillars is odd-graceful. We also show that every tree of diameter up to five is odd-graceful.

Keywords: Odd-graceful labeling, $\alpha$-labeling, trees of diameter 5.

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1. Introduction

Let $G$ be a graph of order $m$ and size $n$, a difference vertex labeling of $G$ is an assignment $f$ of labels to the vertices of $G$ that induces for each edge $xy$ a label or weight given by the absolute value of the difference of its vertex labels. Graceful labelings are a well-known type of difference vertex labeling; a function $f$ is a graceful labeling of a graph $G$ of size $n$ if $f$ is an injection from $V(G)$ to the set $\{0, 1, \ldots, n\}$ such that, when each edge $xy$ of $G$ has assigned the weight $|f(x) - f(y)|$, the resulting weights are distinct; in other words, the set of weights is $\{1, 2, \ldots, n\}$. A graph that admits a graceful labeling is said to be graceful.

When a graceful labeling $f$ of a graph $G$ has the property that there exists an integer $\lambda$ such that for each edge $xy$ of $G$ either $f(x) \leq \lambda < f(y)$ or $f(y) \leq \lambda < f(x)$, $f$ is named an $\alpha$-labeling and $G$ is said to be an $\alpha$-graph. From the definition it is possible to deduce that an $\alpha$-graph is necessarily bipartite and that the number $\lambda$ (called the boundary value of $f$) is the smaller of the two vertex labels that yield the edge with weight 1. Some examples of $\alpha$-graphs are the cycle $C_n$ when $n \equiv 0 \pmod{4}$, the complete bipartite graph $K_{m,n}$, and caterpillars (i.e., any tree with the property that the removal of its end vertices leaves a path).
A little less restrictive than $\alpha$-labelings are the odd-graceful labelings introduced by Gnanajothi in 1991 [4]. A graph $G$ of size $n$ is odd-graceful if there is an injection $f : V(G) \rightarrow \{0, 1, 2, ..., 2n-1\}$ such that the set of induced weights is $\{1, 3, ..., 2n-1\}$. In this case, $f$ is said to be an odd-graceful labeling of $G$. One of the applications of these labelings is that trees of size $n$, with a suitable odd-graceful labeling, can be used to generate cyclic decompositions of the complete bipartite graph $K_{n,n}$. In Figure 1 we show an odd-graceful tree of size 6 together with its embedding in the circular arrangement used to produce the cyclic decomposition of $K_{6,6}$. Once the labeled tree has been embedded, successive $60^\circ$ (counterclockwise) rotations produce the desired cyclic decomposition of $K_{6,6}$.

![Figure 1: Cyclic decomposition of $K_{6,6}$](image)

Gnanajothi [4] proved that the class of odd-graceful graphs lies between the class of $\alpha$-graphs and the class of bipartite graphs; she proved that every $\alpha$-graph is also odd-graceful. The reverse case does not work, for example the odd-graceful tree shown in Figure 1 is the smallest tree without an $\alpha$-labeling. Since many families of $\alpha$-graphs are known, the most attractive examples of odd-graceful graphs are those without an $\alpha$-labeling or where an $\alpha$-labeling is unknown; for instance, Gnanajothi [4] proved that the following are odd-graceful graphs: $C_n$ when $n \equiv 2 \pmod{4}$, the disjoint union of $C_4$, the prism $C_n \times K_2$ if and only if $n$ is even, and trees of diameter 4 among others. Eldergil [2] proved that the one-point union of any number of copies of $C_6$ is odd-graceful. Seoud, Diab, and Elsakhawi [5] showed that a connected $n$-partite graph is odd-graceful if and only if $n = 2$ and that the join of any two connected graphs is not odd-graceful.

A detailed account of results in the subject of graph labelings can be found in Gallian’ survey [3].

Gnanajothi [4] conjectured that all trees are odd-graceful and verified this conjecture for all trees with order up to 10. The author has extended this up to trees with order up to $12^1$. In this paper we prove that all trees of diameter 5 are odd-graceful and that any forest whose components are caterpillars is odd-graceful.

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1Odd-graceful labelings of trees of order 11 and 12 can be found at http://cims.clayton.edu/cbarrien/research
2. Odd-Graceful Forests

In this section we study forests that accept odd-graceful labelings. Recall that a forest with more than one component cannot be graceful because it has "too many vertices". Gnanajothi [4] proved that every \(\alpha\)-graph is odd-graceful. In fact, let \(G\) be an \(\alpha\)-graph of size \(n\). Suppose that \(f\) is an \(\alpha\)-labeling of \(G\) such that \(\max\{f(x) : x \in A\} < \min\{f(x) : x \in B\}\), where \(\{A, B\}\) is the bipartition of \(V(G)\). An odd-graceful labeling of \(G\) is given by

\[
g(x) = \begin{cases} 2f(x), & x \in A \\ 2f(x) - 1, & x \in B. \end{cases}
\]

In Figure 2 we show an example of an \(\alpha\)-labeling of a caterpillar of size 10, together with its corresponding odd-graceful labeling. We use these labelings in the proof of Theorem 1.

**Theorem 1.** Any forest whose components are caterpillars is odd-graceful.

**Proof.** Let \(F_i\) be a caterpillar of size \(n_i \geq 1\), for \(1 \leq i \leq k\). Let \(u_i, v_i \in V(F_i)\) such that \(d(u_i, v_i) = \text{diam}(F_i)\); so identifying \(v_i\) with \(u_{i+1}\), for each \(1 \leq i \leq k - 1\), we have a caterpillar \(F\) of size \(\sum_{i=1}^{k} n_i = n\). Now we proceed to find both, the \(\alpha\)-labeling of \(F\) and its corresponding odd-graceful labeling, using the scheme shown in Figure 2. Once the odd-graceful labeling has been obtained, we disengage each caterpillar \(F_i\) from \(F\), keeping their labels; in this form, the weights induced are \(\{1, 3, \ldots, 2n-1\}\). To eliminate the overlapping of labels we subtract 1 from each vertex label of \(F_i\) when \(i\) is even, in this way the weights remain the same and the labels assigned on \(u_{i+1}\) and \(v_i\) differ by one unit. Therefore, the labeling of the forest \(\bigcup_{i=1}^{k} F_i\) is odd-graceful.

In Figure 3 we show an example of this construction using the odd-graceful labeling obtained in Figure 2.

The procedure used in this proof can be extended to the disjoint union of graphs with \(\alpha\)-labelings. In fact, suppose that the concatenation of blocks \(B_1, B_2, \ldots, B_k\) results in a graph \(G\) whose block-cutpoint graph is a path; in [1] we proved that \(G\) is an \(\alpha\)-graph...
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Figure 3: Odd-graceful labeling of a forest

provided that each $B_i$ is an $\alpha$-graph. Transforming this $\alpha$-labeling into an odd-graceful labeling and disconnecting $G$ into blocks, the disjoint union of these blocks is odd-graceful.

**Theorem 2.** The disjoint union of blocks that accept $\alpha$-labelings is odd-graceful.

As consequence, any forest which components are $\alpha$-trees is odd-graceful.

3. Odd-Graceful Trees of Diameter Five

Every tree of diameter at most 3 is a caterpillar, therefore it is odd-graceful. Gnanajothi [4] proved that every rooted tree of height 2 (that is, diameter 4) is odd-graceful. In the next theorem, we represent trees of diameter 5 as rooted trees of height 3 and prove that they are odd-graceful.

Let $T$ be a tree of diameter 5; $T$ can be represented as a rooted tree of height 3 by using any of its two central vertices as the root vertex. Note that only one of the vertices in level 1 has descendants in level 3; this vertex will be located in the right extreme of level 1. Now, within each level, the vertices are placed from left to right in such a way that their degrees are increasing. In the proof of the next theorem we use this type of representation of $T$, that is, assuming that $v$ (one of the two central vertices) is the root.

**Theorem 3.** All trees of diameter five are odd-graceful.

**Proof.** Let $T$ be a tree of diameter 5 and size $n$. Suppose that $T$ has been drawn according to the previous description. Let $v_{i,j}$ denote the $i$th vertex of level $j$, for $j = 1, 2, 3$; this vertex is placed at the right of $v_{i+1,j}$. Consider the labeling $f$ of the vertices within each level given by recurrence as follows: $f(v) = 0$, where $v$ is the root of $T$, $f(v_{1,1}) = 2n − 2\deg(v) + 1$, $f(v_{1,2}) = 2$, $f(v_{1,3}) = 3$, now the labels are set for the initial vertices of
each level and \( f(v_{i,j}) = f(v_{i-1,j}) + d(v_{i,j}, v_{i-1,j}) \) where \( i \geq 2, 1 \leq j \leq 3 \), and \( d(v_{i,j}, v_{i-1,j}) \) represents the distance between the vertices \( v_{i,j} \) and \( v_{i-1,j} \).

We claim that \( f \) is an odd-graceful labeling of \( T \). In fact, let us see that there is no overlapping of labels. On level 0 the label used is 0 and on level 2 all labels are even, being 2 the smallest label used here. On levels 1 and 3 the labels used are odd; on level 1 the labels used are \( 2n - 1, 2n - 3, ..., 2n - 2 \deg(v) + 1 \), while on level 3 the labels used are 3, 5, .... We want to prove that the largest label on level 3 is less than the smallest label on level 1.

Suppose that \( k \) is the number of vertices on level 3; thus the weights on level 3 edges are \( 1, 3, ..., 2k - 1 \); if \( v_{t,2} \) is the last son of \( v_{1,1} \) that has sons on level 3, then the weight \( 2k - 1 \) must be obtained on the edge \( v_{t,2}v_{k,3} \). Since \( f(v_{k,3}) = f(v_{t,2}) + 2k - 1 \leq 2 \deg(v_{1,1}) + 2k - 3 \), we claim that \( f(v_{k,3}) < f(v_{1,1}) \). In fact, since \( \deg(v) + \deg(v_{1,1}) < n - k + 2 \), we may conclude that \( 2 \deg(v_{1,1}) + 2k - 3 < 2n - 2 \deg(v) - 1 \). Hence, the largest label on level 3 is less than the smallest label on level 1, which implies that there is no overlapping of labels.

As a consequence of the fact that labels used in consecutive levels have different parity, each weight obtained is an odd number not exceeding \( 2n - 1 \). Suppose that \( v_{i+1,j} \) and \( v_{i,j} \) have the same father \( x \), by definition of \( f \), the edges \( xv_{i+1,j} \) and \( xv_{i,j} \) have consecutive weights. If \( v_{i+1,j} \) and \( v_{i,j} \) have different father, \( x \) and \( y \), respectively, then \( |f(y) - f(v_{i,j})| = |(f(x) + 2) - (f(v_{i+1,j}) + 4)| = |f(x) - f(v_{i+1,j}) - 2| \). Thus, on level 1 the weights are \( 2n - 1, ..., 2n - 2 \deg(v) + 1 \), on level 2, the weights are \( 2n - 2 \deg(v) - 1, ..., 2k + 1 \), and on level 3 the weights are \( 2k - 1, ..., 1 \).

Therefore, \( f \) is an odd-graceful labeling of \( T \).

In Figure 4 we present a scheme of this labeling for a tree of size 23.

![Figure 4: Odd-graceful tree of diameter 5](image-url)

Similar arguments can be used to find odd-graceful labelings of trees of diameter 6; however we do not have a general labeling scheme for this case. So it is an open problem...
determining whether trees of diameter 6 are odd-graceful. In Figure 5, we give an example of an odd-graceful labeling for a tree of size 17 and diameter 6.

To conclude this section, we show in Figure 6 an odd-graceful labeling for a special type of tree of diameter 8, namely the star $S(n, 4)$ with $n$ spokes of length 4. The deletion of the vertices in the last row produces the star $S(n, 3)$, a graceful labeling of this tree is obtained by subtracting $2n$ from the labels on the odd-numbered levels.

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