Graph Labelings for Everyone!

Christopher Raridan

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I have been very lucky to work with a lot of wonderful professors and students on research projects since arriving at Clayton State University in 2008.

Without their support and encouragement, I doubt I would be standing here right now.
Introduction
Graph Labelings
Super Edge-Graceful Labelings (SEGL)
Edge-Balanced Index Sets (EBI)

Basic Graph Theory

- Graph, \( G = (V, E) \), a pair of sets
- Vertex set \( V \) (some dots/nodes)
- Edge set \( E \) (connect the dots)
- Order = \(|V|\) and Size = \(|E|\)
Basic Graph Theory

- Vertices $v_1$ and $v_3$ are incident with edge $v_1v_3$.

- Vertices $v_1$ and $v_3$ are adjacent.

- The neighborhood of $v_1$ is $N(v_1) = \{v_2, v_3, v_5\}$.

- The degree of $v_1$ is $|N(v_1)| = 3$.

- If every vertex has the same degree, the graph is regular.
Gallian’s *A Dynamic Survey of Graph Labeling*

- First draft: Nov. 14, 1997
- Current version: Dec. 7, 2015 (18 editions)
- Online and free to access!
- Organizes results for 200 graph labelings from 2000 papers
- 5 decades of graph labeling!
  - few general results
  - most results only for classes of graphs
  - such diversity means some problems have been addressed multiple times using different terminology
What’s a Graph Labeling?

An assignment of integers to the vertices or edges of a graph:

- A set of integers from which vertex labels are chosen
- A set of integers from which edge labels are chosen
- A rule that assigns a value to each vertex or edge
- Others conditions which must be met
In the beginning... Rosa created “valuations of the vertices of a graph.” (1967)

- $f : V \rightarrow \{0, 1, \ldots, |E|\}$ is one-to-one
- $f^* : E \rightarrow \{1, \ldots, |E|\}$ is one-to-one & onto
- $f^*(uv) = |f(u) - f(v)|$
- $\beta$-valuation, later “graceful” by Golumb (1972)
Graceful Example

- \( f : V \rightarrow \{0, \ldots, 6\} \)

- \( f^* : E \rightarrow \{1, \ldots, 6\} \)

- Erdős (~1972): Most graphs are not graceful.
Harmonious Labelings

Graham & Sloane (1980) defined a harmonious labeling of a graph.

- $\mathbb{Z}_n = \{0, 1, \ldots, n - 1\}$
- $f : V \rightarrow \mathbb{Z}_{|E|}$ is one-to-one†
- $f^* : E \rightarrow \mathbb{Z}_{|E|}$ is one-to-one & onto
- $f^*(uv) = \left( f(u) + f(v) \right) \pmod{|E|}$
- †For trees, two vertices get same label
Harmonious Example

- 6 edges $\Rightarrow$ addition is modulo 6
- $f : V \rightarrow \{0, \ldots, 5\}$
- $f^* : E \rightarrow \{0, \ldots, 5\}$
- (G&S, 1980) Most graphs are not harmonious.
Trees

A tree on \( n \) vertices is a connected acyclic graph with \( n - 1 \) edges.

A **caterpillar** is a tree in which removal of the "leaves" results in a path.

![Caterpillar Example](image1)

\[ RT(0^4, 2, 3) \]

A **lobster** is a tree in which removal of the leaves results in a caterpillar.

![Lobster Example](image2)

\[ RT(0, 2^2, 3) \]
Graceful Tree Conjecture

Conjecture

(Ringel & Kotzig, 1982) All trees are graceful.

Theorem

(Aldred & McKay, 1998) All trees with at most 27 vertices are graceful.

Theorem

(Fang, 2010, Tree Verification Project) All trees with at most 35 vertices are graceful.
Harmonious Tree Conjecture

Conjecture

(Graham & Sloane, 1980) All trees are harmonious.

Theorem

(Aldred & McKay, 1998) All trees with at most 26 vertices are harmonious.

Theorem

(Fang, 2014) All trees with at most 31 vertices are harmonious.
Quick Tour of SEGL

- **1985**: Lo, *On edge-graceful labelings of graphs*

- **1994**: Mitchem and Simoson, *On edge-graceful and super edge-graceful labelings of graphs*

- **2006**: Chung, Lee, Gao, and Schaffer, *On the super edge-graceful trees of even orders*,

  Characterize trees of diameter 4 which are super edge-graceful.

- **2007**: Lee and Ho, *All trees of odd order with three even vertices are super edge-graceful*
Quick Tour of SEGL

- **2008**: Gao and Zhang, *A note on the super edge-graceful labelings of caterpillars*

- **2009**: Cichacz, Froncek, Khodkar, and Xu, *Super edge-graceful paths and cycles*

- **2011**: Krop, Mutiso, and R., *On super edge-graceful trees of diameter four*

- **2015**: Collins, Magnant, and Wang, *Tight super edge-graceful labelings of trees and their applications*
Super Edge-Graceful Labelings

**SEGL**: A bijection $f$ from $E$ to

$$\{0, \pm 1, \pm 2, \ldots, \pm (|E| - 1)/2\}, \quad \text{if } |E| \text{ is odd},$$

$$\{\pm 1, \pm 2, \ldots, \pm |E|/2\}, \quad \text{if } |E| \text{ is even},$$

that induces a bijection $f^+$ from $V$ to

$$\{0, \pm 1, \pm 2, \ldots, \pm (|V| - 1)/2\}, \quad \text{if } |V| \text{ is odd},$$

$$\{\pm 1, \pm 2, \ldots, \pm |V|/2\}, \quad \text{if } |V| \text{ is even},$$

where $f^+(v) = \sum_{uv \in E} f(uv)$. 

Raridan

Graph Labelings!
Results for Caterpillars

Theorem

All caterpillars of diameter 4 with even size are super edge-graceful.

Theorem

All caterpillars of diameter 4 with odd size are super edge-graceful, except $RT(0^j, 1, b_{j+2})$ when $j, b_{j+2} \geq 1$ are odd with $j = 1$ or $b_{j+2} = 1$. 
SEG Caterpillar Example

\(RT(0^3, 3, 5)\) is SEG.

Edge labels: \(\{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6\}\)

Vertex labels: \(\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7\}\)
Results for Lobsters

Theorem

All lobsters of diameter 4 with even size are super edge-graceful.

Unfortunately, we were not able to find general results for all odd size lobsters of diameter 4.

We were able to provide SEGL for several families of odd lobsters and we determined that one family was not SEG.
SEG Lobster Example

$RT(0^4, 2^2, 3)$ is SEG.

Edge labels: \{±1, ±2, ±3, ±4, ±5, ±6, ±7\}

Vertex labels: \{0, ±1, ±2, ±3, ±4, ±5, ±6, ±7\}
Quick Tour of EBI

- **2009:** Kong, Wang, and Lee, *On edge-balanced index sets of some complete k-partite graphs*

- **2012:** Krop, Lee, and R., *On the edge-balanced index sets of product graphs*

- **2013:** Krop, Minion, Patel, and R., *A solution to the edge-balanced index set problem for complete odd bipartite graphs*
Quick Tour of EBI

- **2014**: Hua and R., *On the edge-balanced index sets of odd/even complete bipartite graphs*

- **2016**: Dao, Hua, Ngo, and R., *On the edge-balanced index sets of complete even bipartite graphs*

- **2016/7**: Dao and Hua, *On the edge-balanced index sets of even/odd complete bipartite graphs*, in preparation
Complete Bipartite Graphs

This is $K_{5,3}$.


- **edge-labeling**, $f : E(G) \rightarrow \{0, 1\}$
- **1-edge** = edge labeled 1
- **0-edge** = edge labeled 0
- **$e(1)$** = number of 1-edges
- **$e(0)$** = number of 0-edges
- **edge-friendly**, $|e(1) - e(0)| \leq 1$
Partial Vertex Labeling

- \( \text{deg}_1(v) = \) 1-degree of \( v \)
- \( \text{deg}_0(v) = \) 0-degree of \( v \)
- partial vertex-labeling,

\[
f^*(v) = \begin{cases} 
1, & \text{if } \text{deg}_1(v) > \text{deg}_0(v) \\
0, & \text{if } \text{deg}_0(v) > \text{deg}_1(v) 
\end{cases}
\]

- if \( \text{deg}_1(v) = \text{deg}_0(v) \), then
  \( f^*(v) \) is undefined

\( K_{3,2} \)
**Edge-Balanced Index Set**

- 1-vertex = vertex labeled 1
- 0-vertex = vertex labeled 0
- unlabeled vertex has no label
- \( v(1) \) = number of 1-vertices
- \( v(0) \) = number of 0-vertices

\[
EBI(G) = \{ |v(1) - v(0)| : \text{ over ALL edge-friendly labelings} \}
\]
This example shows that
\[ 0 \in EBI(K_{3,2}). \]

Are there any other values in \( EBI(K_{3,2}) \)?
Motivating Questions

**Big Question:** How do we find ALL the elements in $EBI(K_{m,n})$?

**Easier Question:** Can we show $0 \in EBI(K_{m,n})$?

**Important Question:** How do we find $\max EBI(K_{m,n})$?
Thank you!

Christopher Raridan ..... ChristopherRaridan@clayton.edu

Draft versions of most papers I’ve coauthored can be found on arXiv.org or ResearchGate.com.
A Favor to Ask. . .

Are you interested in helping to judge posters Saturday morning?

10:10 – 10:45am, HSC 3rd floor lobby