Edge-Balanced Index Sets of Complete Bipartite Graphs

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93rd MAA-SE

Saturday, March 15, 2014
Abstract

In this presentation, we will

- provide definitions related to the EBI problem.
- discuss some history of the EBI problem.
- give the EBI of odd complete bipartite graphs.
- discuss the EBI problem for complete bipartite graphs where at least one part has even cardinality.
Edge-Friendly Labeling

- Graph $G = (V, E)$.
- All edges labeled either 0 or 1.
- Edge-friendly labeling: absolute difference in number of 0-edges and 1-edges $\leq 1$. 

![Graph Diagram]
(Partial) Vertex Labeling

- Edge-friendly labeling $f$ induces a (partial) vertex labeling $f^+$.

- $\text{deg}_0(v) > \text{deg}_1(v) \Rightarrow f^+(v) = 0$ and $v$ is a 0-vertex.

- $\text{deg}_1(v) > \text{deg}_0(v) \Rightarrow f^+(v) = 1$ and $v$ is a 1-vertex.

- $\text{deg}_1(v) = \text{deg}_0(v) \Rightarrow f^+(v)$ is undefined and $v$ is unlabeled.


- $F$ is the set of all edge-friendly labelings of $G$ and $f \in F$.
- $v_f(i)$ is number of $i$-vertices.
- **Edge-balanced index set** of $G$:

$$EBI(G) = \{|v_f(0) - v_f(1)| : \forall f \in F\}$$

"This" $f$ gives $|v_f(0) - v_f(1)| = 1$.
"This" $f$ gives $|v_f(0) - v_f(1)| = 0$. 

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*EBI of Complete Bipartite Graphs*
A Quick Tour

Much work has been done in classifying graphs based on $EBI$:

- **1992**: Lee, Liu, and Tan, *On balanced graphs*

- **1995**: Kong and Lee, *On edge-balanced graphs*

- **2002**: Chen, Huang, Lee and Liu, *On edge-balanced multigraphs*
A Quick Tour

- **2009**: Kong, Wang, and Lee, *On edge-balanced index sets of some complete k-partite graphs*

- **2012**: Krop, Lee, and R., *On the edge-balanced index sets of product graphs*

- **2013**: Krop, Minion, Patel, and R., *A solution to the edge-balanced index set problem for complete odd bipartite graphs*
Find the edge-balanced index set for a family of graphs.

The Standard Approach:

1. Find the maximal element in the set.

2. Provide an algorithm to produce the lesser elements.

Step 2 could be replaced by “show that the lesser elements exist” without providing a specific algorithm.
The First Results and a Nice Theorem

**KW L, 2009:** \( EBI(K_{m,n}) \) for \( n = 1, 2, 3, 4, 5, \) and \( m = n. \)

**Theorem**

(KMPR, 2013*) Let \( K_{m,n} \) be a complete bipartite graph with parts of cardinality \( m \geq n \geq 1, \) where \( m, n \) are both odd, and let \( k = \left\lceil \frac{m-1}{n+1} \right\rceil. \) Then

\[
EBI(K_{m,n}) = \begin{cases} 
\{2\}, & \text{if } n = 1, \\
\{0, 2, \ldots, m + n - 2k - 2\}, & \text{otherwise.}
\end{cases}
\]

*Article to appear in next issue of Bull. ICA.*
EBI($K_{m,n}$) for Even $m$ or $n$

**Theorem**

*(HKR, 2014)* Let $K_{m,n}$ be a complete bipartite graph with parts of cardinality $m$ and $n$, where $m$ is odd, $n$ is even, and $m > n \geq 2$. Then $EBI(K_{m,2}) = \{0\}$. For $n \geq 4$, let $q$ be the quotient when $m$ is divided by $\frac{n}{2} + 1$ and let $r$ be the remainder. Then

$$EBI(K_{m,n}) = \begin{cases} 
\{0, 1, \ldots, m + n - 2q - 2\}, & \text{if } r = 0, \\
\{0, 1, \ldots, m + n - 2q - 3\}, & \text{if } r = 1, \\
\{0, 1, \ldots, m + n - 2q - 4\}, & \text{if } r \geq 2.
\end{cases}$$
Sketch of Proof

Want edge-friendly labeling of $K_{m,n}$.

All vertices in part A are unlabeled.

All but one of the vertices in part B are 1-vertices and other is 0-vertex.

Then $n - 2 \in EBI(K_{m,n})$.

Pick “special” vertex $v$ in each of the $q$ groups of $\frac{n}{2} + 1$ vertices in part A.
For each group and $2 \leq i \leq \frac{n}{2} + 1$, label edge $vu_i$ for by 1.

For each group, label perfect matching among remaining vertices and $u_i$ by 0.

Label all remaining edges to obtain desired edge-friendly labeling.
For each group, start switching labels on 0-edges from perfect matching with labels on 1-edges $vu_i$.

First switch does not increase current index.

Each successive switch will increase index by 1.

Total increase of $q \left( \frac{n}{2} - 1 \right)$. 
Sketch of Proof (cont.)

We deal with remaining $r$ unlabeled vertices in part A similarly.

If $r = 0$, then $v(1) = (n - 1) + (m - q)$, $v(0) = 1 + q$, and

$$\max\{EBI(K_{m,n})\} = v(1) - v(0) = m + n - 2q - 2.$$ 

If $r = 1$, then $v(1) = (n - 1) + (m - q - 1)$, $v(0) = 1 + q$, and

$$\max\{EBI(K_{m,n})\} = m + n - 2q - 3.$$ 

If $r \geq 2$, then $v(1) = (n - 1) + (m - q - 1)$, $v(0) = 2 + q$, and

$$\max\{EBI(K_{m,n})\} = m + n - 2q - 4.$$
Note that we have only shown that the indices from $n - 2$ to $\max\{\text{EBI}(K_m,n)\}$ are in $\text{EBI}(K_m,n)$.

We use a similar argument to show that the indices from 0 to $n - 3$ are in $\text{EBI}(K_m,n)$. 
Thank you!

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